Making maps compatible with GPS

Transformations between The Irish Grid
and the GPS Co-ordinate Reference Frame
WGS84 / ETRF89

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CONTENTS

INTRODUCTION

The Global Positioning System (GPS) allows a user with an appropriate receiver to obtain their position anywhere on or above the earth's surface. Although originally designed for military use, a large number of civilian applications have developed in recent years. GPS positions are based on a Global reference system that was originally defined to an accuracy of 1 metre. This reference system has been improved and refined in many regions of the Globe - it is now possible to obtain sub-centimetre levels of accuracy, if the right equipment and techniques are employed.

The Irish Grid is the "framework" on which maps in Ireland are hung. All positions on Irish mapping are the result of a comprehensive series of observations carried out at triangulation stations throughout Ireland during the 1950's and 1960's. These observations used terrestrial based systems, such as theodolites and Electronic Distance Measuring devices. All the observations were combined into one mathematical computation for the whole of Ireland. This resulted in a solution for the positions of the triangulation stations, known as the Ireland 1975 (Mapping) Adjustment.

The GPS reference system and the Irish Grid have been derived by different methods, on different datums, within different reference frames and with positions expressed on different co-ordinate systems. Therefore, GPS positions are not directly compatible with the Irish Grid and they must be transformed in order to relate correctly.

This technical booklet describes how to convert Irish Grid positions to GPS positions, and vice versa. It is the second in a series of technical papers aimed at informing map users and the public in general alike on a number of technical matters.

Following a general overview of co-ordinate reference systems, the GPS reference system and the Irish Grid are briefly described. The approaches taken to transform GPS to Irish Grid positions are then outlined. A description of two of these methods available are then provided, with worked examples.

Acknowledgements

We acknowledge the work of the Institute of Engineering Surveying & Satellite Geodesy (IESSG) at the University of Nottingham who developed the transformation under contract for Ordnance Survey Ireland and the Ordnance Survey of Northern Ireland.

GEODETIC CO-ORDINATE REFERENCE SYSTEMS

Introduction

Positions may be expressed in one of three basic forms, 3D earth centred Cartesian, geographical or plane co-ordinates. Each form is dependent upon a reference system, and positions will differ for different reference systems. This section introduces the basic concepts of Cartesian, Geographical and Plane co-ordinates. Transformation between reference systems is introduced, and finally the GPS reference system and the Irish Grid reference system are outlined.

Cartesian Co-ordinates

Positions may be given in absolute terms, relative to the earth's centre of mass, or an assumed centre (as implied by a geodetic datum). In this system a position is defined in 3 dimensional space by an X, Y, Z co-ordinate triplet. The Z axis passes through the centre of the earth (or reference ellipsoid) and the poles, the X axis through the centre and the Greenwich meridian, and the Y axis at right angles to these. Other parameters may define this system, but are not directly relevant here.

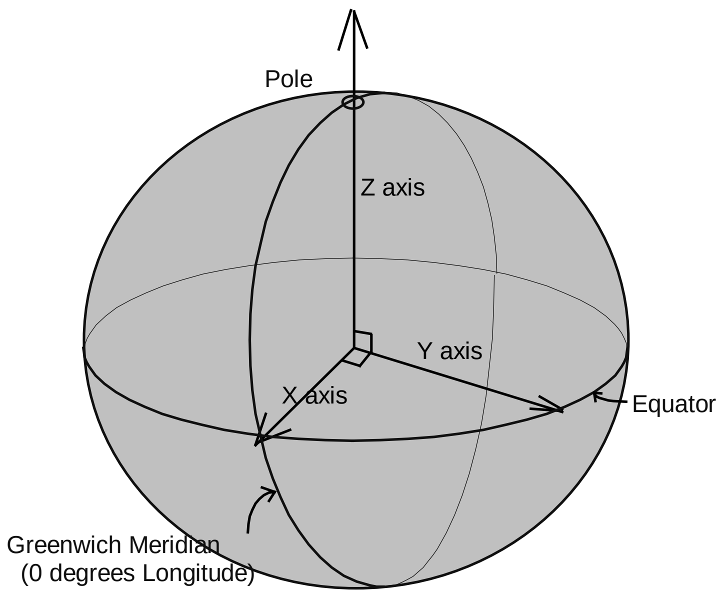

Diagram 1 ***: Cartesian Co-ordinates.***

If the 3-dimensional earth centred co-ordinate system is associated with a reference ellipsoid, positions may be transformed to or from geographical co-ordinates, using standard mathematical formulae (Reference [3] & [6]).

Geographical Co-ordinates

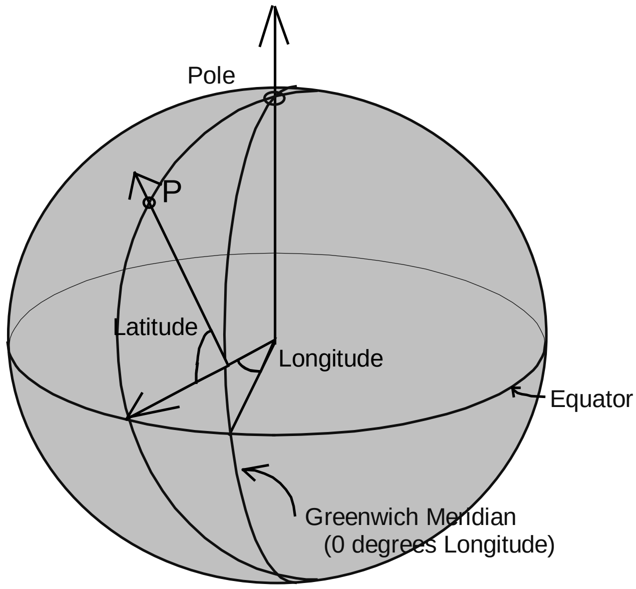

Diagram 2 : ***Geographical Co-ordinates.***

A position P on the earth's surface can be defined in terms of it's latitude, ϕ, longitude, λ, and height, h (height above the reference ellipsoid). The expression of such a position requires a reference ellipsoid. This reference ellipsoid is the nearest simple mathematical shape to the shape of the earth. The size, shape and position of the reference ellipsoid are dependent upon the extent of the area to be mapped, and the technology used to determine the shape of the earth in that area. The defining parameters of the reference ellipsoid, its position and orientation are known as the geodetic datum.

Plane Co-ordinates

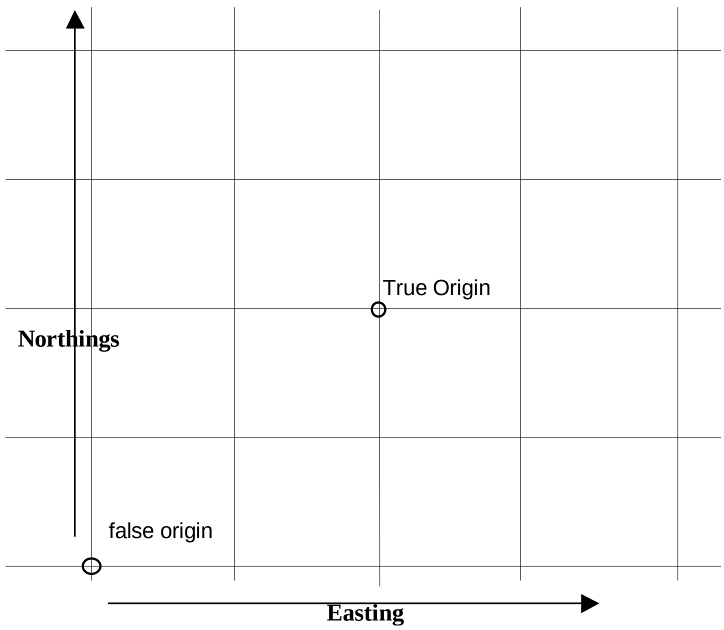

Diagram 3 : ***Plane Co-ordinates***

It is usual to depict features from the earth's surface on paper, or some other two dimensional medium. This is achieved by mathematically projecting geographical co-ordinates onto a plane [1]. Positions can then be expressed in terms of eastings and northings.

- Eastings are the distance, in metres, in an easterly direction from some origin.
- Northings are the distance, in metres, in a northerly direction from some origin.

Transformation between Geodetic Datum

The three co-ordinate forms described above are each dependent upon a Geodetic Datum. Thus the description of the position of a point or set of points should include the reference system used. Because of the possibility of two different geodetic datum, one physical point on the earth's surface could have two different co-ordinates. Therefore it is necessary to relate the two by some mathematical transformation in order to express co-ordinates in the same system.

A number of different procedures are available for performing co-ordinate transformations between different reference systems. The advantages and disadvantages of each procedure must be considered in terms of the systems' to be modelled, the accuracy required, and the simplicity of application. There are three basic forms of transformation procedure commonly used; Plane [2], Helmert or Molodenskii [4] and Multiple Regression [4].

Irish Grid Reference System

The Irish Grid is a plane co-ordinate system based on a modified Transverse Mercator Projection. Map positions expressed in this system are based on a co-ordinate reference frame observed by two primary triangulation's during the 1950's and 60's, and combined in one adjustment in 1975 to produce geographic positions (latitude and longitude) for the primary stations in the reference frame. This adjustment is known as the 1975 (Mapping) Adjustment. A modified Airy ellipsoid was used as the figure for the earth. The Geodetic Datum is known as the 1965 Datum, and is defined by the positions of the ten Northern Ireland primaries (as defined by the 1952 adjustment) and the positions of two primary stations in the Republic (as defined by the 1965 adjustment).

Geographic co-ordinates (ϕ, λ) were projected onto the plane grid using standard Transverse Mercator projection formulae with Irish parameters [1]. Secondary and tertiary triangulation's, traverses and mapping control subsequently established Irish Grid (Eastings and Northings) positions relative to this co-ordinate frame, from which all mapping have been based since the mid 1970's.

Positions on maps are expressed in two dimensions as Eastings (E) and Northings (N) relative to a false origin. Re-projection of two dimensional grid co-ordinates (E,N) back into geographic or ellipsoidal co-ordinates (ϕ, λ) are possible by using standard formulae [1].

Recent measurement's have confirmed the consistency of positions within the network as generally better than 25 cm's (between adjacent stations) which confirms the quality of the Irish Reference system. However, modern measurement techniques use a global reference system, and these indicate that the absolute accuracy of the Irish Reference framework is everywhere better than 1 m.

GPS reference system

WGS84 and GRS80

The adoption of regional or local reference ellipsoids results in different positions for the same point along common boundaries between two different regions. The development of a global reference system was largely driven by international military requirements, and resulted in the World Geodetic System, WGS84. WGS84 is a geocentric reference ellipsoid and a geodetic datum, in that it defines the centre of mass of the earth as its origin, and the direction of the earth's axis as the minor axis of the reference ellipsoid. GRS80 is a further refinement of the WGS84, and is coincident with WGS84 at the metre level. Although WGS84 was originally defined to a precision of 1 m in any axis, more precise reference systems have been defined internationally (such as ITRF) and regionally (such as ETRF) using the GRS80 ellipsoid and high precision satellite observations at global and regional geodetic observation facilities.

ETRS89

The high precision geodetic global reference frame is known as the International Terrestrial Reference Frame (ITRF), with positions expressed as three dimensional earth centred Cartesian co-ordinates. Geographic co-ordinates (latitudes and longitudes) are based on the GRS80 ellipsoid as a best fitting global figure for the earth. Within Europe a network of permanently recording geodetic facilities have enabled a precise subset of the ITRF to be established, known as the European Terrestrial Reference Framework (ETRF). Due to the precision of modern measurements, movements of stations have been detected between sets of observations taken at different times and may result in inconsistency of positions observed at different times over a few years duration. Thus different co-ordinate systems are established, within the defining reference frame, linked to the epoch of the observation, and transformations have been defined between them, which move positions from those determined at the time of observations to a common system at a defined epoch. The adopted reference system for Europe is known as the European Terrestrial Reference System, 1989 (ETRS89).

Co-ordinates in ETRS89 are expressed as either three dimensional (X, Y, Z) Cartesian co-ordinates or as three dimensional ellipsoidal co-ordinates (ϕ, λ and H, Ellipsoidal height), based on the GRS80 ellipsoid.

IRENET95

During 1995 the Ordnance Surveys' in Ireland established a network of 173 stations, called IRENET, throughout Ireland which have positions computed in terms of ETRF89. Results indicate an accuracy of better than 20 mm overall, and a precision of only a few mm's. Thus, in terms of mapping in Ireland, it may be considered distortion free.

During the IRENET95 campaign observations were also taken to connect some of the IRENET stations to stations with Irish Grid positions. Therefore these stations have Ireland 1975 (Mapping) Adjustment co-ordinates.

RELATING GPS AND MAPPING REFERENCE SYSTEMS

Comparisons

IRENET 95 stations have been connected to the pre-existing mapping framework known as the Irish Grid. This enabled a comparison to be made between the size and position of Ireland, as defined from the older terrestrial based techniques with Ireland as measured today using satellite based GPS techniques.

Results [5] appear to indicate that Ireland is longer and wider than the previous measurement. In addition, the positions of points between reference systems appear to have shifted by an average of 54.3 metres to the NW as depicted in Diagram 4.

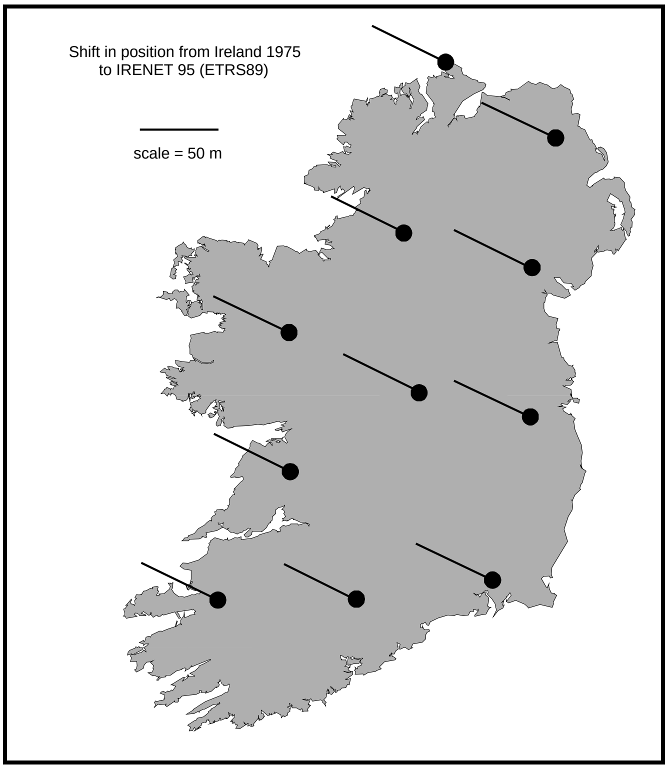

Diagram 4 : ***Shift in position from Ireland 1975 to IRENET 95.***

Why the difference?

The major causes for the apparent movement are due to an improved understanding of the size, shape and position of the Earth, and the manner in which measurements of it are made.

In 1975 the position of the Earth's centre and the direction of its poles was implied by the adoption of the position of certain reference points on its surface, and the orientation between at least two such reference points. Within Ireland this was the nominal position of a station on

Slieve Donard, and the azimuth between the stations on Slieve Donard and Cuilcagh. The positions of these stations are, in effect, those originally determined in the 19th Century. The measurements used in the 1975 mapping adjustment were reduced to a mathematical reference surface known as the Airy Modified Ellipsoid. These parameters define the size, shape and position of a mathematical figure for the Earth in the vicinity of Ireland.

GPS observations are based on a different mathematical figure for the Earth, known as the GRS80 ellipsoid (also termed WGS84). This reference surface was chosen to give a best mean fit over the whole globe. As a consequence, the global figure may not fit as closely to the shape at one location. The GRS80 figure of the Earth is also aligned with the direction of the Earth's axis of rotation. The differences highlighted above are largely due to different figures for the Earth used by the two systems in the vicinity of Ireland. See Diagram 5.

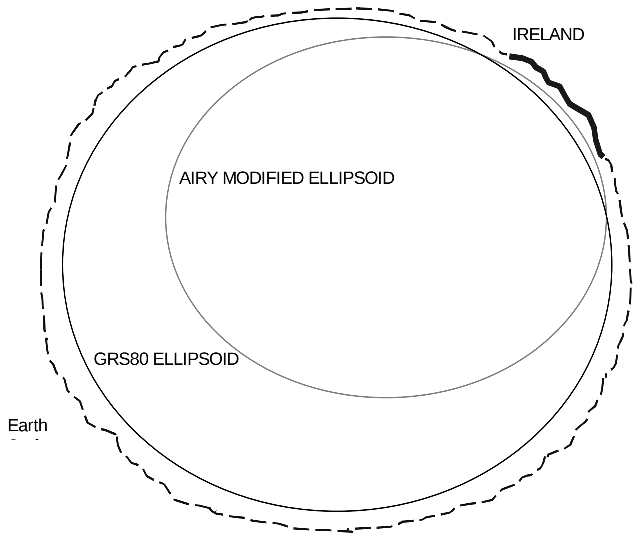

Diagram 5 : ***Exaggerated diagram of ellipsoids and earth surface.***

An additional, but smaller effect, is due to the different type of measurement technologies used. GPS is an order of magnitude more accurate than traditional, terrestrial techniques, and will therefore highlight any distortions in the original network.

To maximise the use of GPS for navigation and scientific work, it must relate to Irish Mapping. IRENET 95 allows the development of a series of "transformations" between the GPS and the Irish mapping reference system that will model the differences between the two. The transformation algorithm to be used will depend on the accuracy required.

Relating GPS to Irish Maps.

In order to relate the two systems we must relate the two figures for the Earth, and also attempt to model some of the distortions in the old system, if they are of a significant size. The level of sophistication required of the algorithm depends upon the accuracy required. 0.1mm at a map scale of 1:50 000 relates to 5m on the ground, implying an accuracy requirement of ±5m. At a map scale of say 1:1 000, 0.1mm is 0.1 metre on the ground, therefore a more precise transformation is required.

Various levels of transformation have been developed and two are presented here.

- Level 1 Transformation (Easting and Northing Shifts)
- Level 2 Transformation (7 parameter Helmert)

For more precise transformations users should contact the relevant Ordnance Survey, Dublin or Belfast for advice.

To maintain the reversibility of the transformations – agreement when transforming from one system to another and back again – it is important that the same level of transformation is used in both directions (Forward Case and Reverse Case).

LEVEL 1 TRANSFORMATION (EASTING AND NORTHING SHIFTS)

Due to the quality of the Ireland 1975 terrestrial network a simple shift of eastings and northings provide an adequate solution for most general purposes.

Derivation

Latitude and Longitude GPS positions (on the GRS80 reference ellipsoid) at the eleven zero order stations were projected into Eastings and Northings using the Transverse Mercator projection, with Irish Grid parameters. These positions may be termed GPS (Irish Grid).

Easting and Northing Irish Grid positions at these eleven stations were then subtracted from GPS (Irish Grid) positions, and a mean shift calculated, to give a difference in Easting (dE) and a difference in Northing (dN) from ETRF89 (GPS) to Ireland 1975 (Irish Grid) as follows:

dE +49.0 m to the Easting

dN -23.4 m to the Northing

These shifts were applied at 159 GPS points throughout Ireland, and the result compared with their known Irish Grid positions. The residuals are depicted in Table 1. The level of accuracy achieved is such that 95% of points transformed in this way are likely to be within 2 metres of their position in the IRENET95 reference frame.

Simple Transformation
GPS (ETRF89) to Map (Ireland 1975)

	dE (m)	dN (m)	Vector (m)
Shift	**49.0**	**-23.4**	**54.3**
Residuals at 159 points =			
Mean	0.0	0.0	0.8
Stnd Dev	0.5	0.2	0.1
RMSE	0.7	0.5	0.8
Max	1.3	1.0	1.5
1.96 x RMSE (95%)=			1.6

Table 1: Residuals after applying simple shift in Easting and Northing.

This approach does not model any rotations or scale change between the two systems. However, as these are everywhere tested less than 1.5 m they are not significant given the required accuracy. Because of the simplicity of this approach over others, it is the preferred "simple" approach. The only requirement is for latitude and longitude GPS positions (on WGS84 or GRS80 ellipsoids) to be projected using the Transverse Mercator and Irish Grid parameters [1]. It should be noted that application of the transformation will not alter the accuracy of the mapping.

Transformation Procedure

In this paper for the implementation of the easting and northing shift transformation, ‘system 1’ is defined as Ireland 1975, and ‘system 2’ is defined as ETRF89, so that a ‘forward’ transformation from system 1 to system 2 converts from Ireland 1975 to ETRF89, and a reverse transformation from system 2 to system 1 converts from ETRF89 to Ireland 1975 as described in the following sections.

Forward Transformation Procedure

The forward transformation procedure is used when transforming from Irish Grid co-ordinates ($Easting_1$, $Northing_1$) to ETRF89 Geodetic Ellipsoidal co-ordinates ($Latitude_2$, $Longitude_2$) and involves the following steps:

- STEP 1: Irish Grid Co-ordinates converted to GPS (Irish Grid) Co-ordinates
- STEP 2: GPS (Irish Grid) converted to ETRF89 Geodetic Ellipsoidal Co-ordinates

STEP 1: Irish Grid Co-ordinates converted to GPS (Irish Grid) Co-ordinates

Shifts of –49.0 metres in the easting and +23.4 metres in the northing are applied to Irish Grid Co-ordinates ($Easting_1$, $Northing_1$) to give GPS (Irish Grid) co-ordinates.

STEP 2: GPS (Irish Grid) converted to ETRF89 Geodetic Ellipsoidal Co-ordinates

GPS (Irish Grid) co-ordinates are converted to ETRF89 geodetic ellipsoidal co-ordinates ($Latitude_2$, $Longitude_2$) using the GRS80 reference ellipsoid and standard Irish Grid parameters. For equations and computational method please refer to Reference [1].

Reverse Transformation Procedure

The reverse transformation procedure is used when transforming from ETRF89 Geodetic Ellipsoidal co-ordinates ($Latitude_2$, $Longitude_2$) to Irish Grid co-ordinates ($Easting_1$, $Northing_1$) and involves the following steps:

- STEP 1: ETRF89 Geodetic Ellipsoidal Co-ordinates projected to GPS (Irish Grid)
- STEP 2: GPS (Irish Grid) Co-ordinates converted to Irish Grid Co-ordinates

STEP 1: ETRF89 Geodetic Ellipsoidal Co-ordinates projected to GPS (Irish Grid)

ETRF89 Geodetic Ellipsoidal co-ordinates ($Latitude_2$, $Longitude_2$ on the GRS80 reference ellipsoid) are projected into Eastings and Northings using the Transverse Mercator projection, with Irish Grid parameters to give GPS (Irish Grid) co-ordinates. For equations and computational method please refer to Reference [1].

STEP 2: GPS (Irish Grid) Co-ordinates converted to Irish Grid Co-ordinates

Shifts of +49.0 metres in the easting and -23.4 metres in the northing are applied to GPS (Irish Grid) Co-ordinates to give Irish Grid co-ordinates ($Easting_1$, $Northing_1$).

LEVEL 2 TRANSFORMATION (HELMERT 7 PARAMETER)

Introduction

The Helmert transformation is one of the most common approaches for the transformation of co-ordinates from one reference system to another, which involves up to 7 systematic biases, namely 3 translations, 3 rotations and a scale factor. This transformation is carried out between two 3-dimensional Cartesian reference systems and therefore the co-ordinates of positions should be presented in (X,Y,Z) format.

As no unified projection method for the ETRF89 co-ordinate reference system has yet been adopted, this paper considers the ETRF89 geodetic ellipsoidal co-ordinates as the output and input co-ordinates for the described procedures.

It is the policy of Ordnance Survey Ireland and the Ordnance Survey of Northern Ireland to supply all GPS manufacturers with the Level 2 Helmert 7 parameter transformation for implementation in both hardware and software. At the time of this booklet going to print the following GPS manufacturers and Agents were identified and supplied with the Level 2 transformation (Appendix D).

Transformation Criteria

In the derivation of this Helmert transformation the following criteria were adhered to:

Accuracy: The transformation should remove datum differences and model the major distortions in the Ireland 1965 datum to an accuracy of ±40 cm. i.e. 95% of a set of transformed Ireland 1975 points should fall within 40 cm of their known ETRF89 co-ordinates, and *vice versa*.

Invertability / Reversibility: Any transformation derived should be capable of being used in the forward case (Ireland 1975 to ETRF89) and in the reverse case (ETRF89 to Ireland 1975).

Uniqueness: Any transformation derived should not allow the possibility of two transformed results for any given data point.

Conformality: The transformation should not distort shapes, such as buildings and boundaries, when transforming them from Ireland 1975 to ETRF89.

Extensibility: The area over which the transformation can be used should extend beyond the area over which it was derived. Specifically, the transformation should be derived from data points on land, and it will be used for some offshore applications.

Method of Parameter Computation

For the derivation of the Helmert transformation co-ordinates in both Ireland 1975 and ETRF89 were provided by the Ordnance Surveys of Ireland and Northern Ireland (as shown in Diagram 6). The co-ordinates of these stations were provided in the form of latitude,

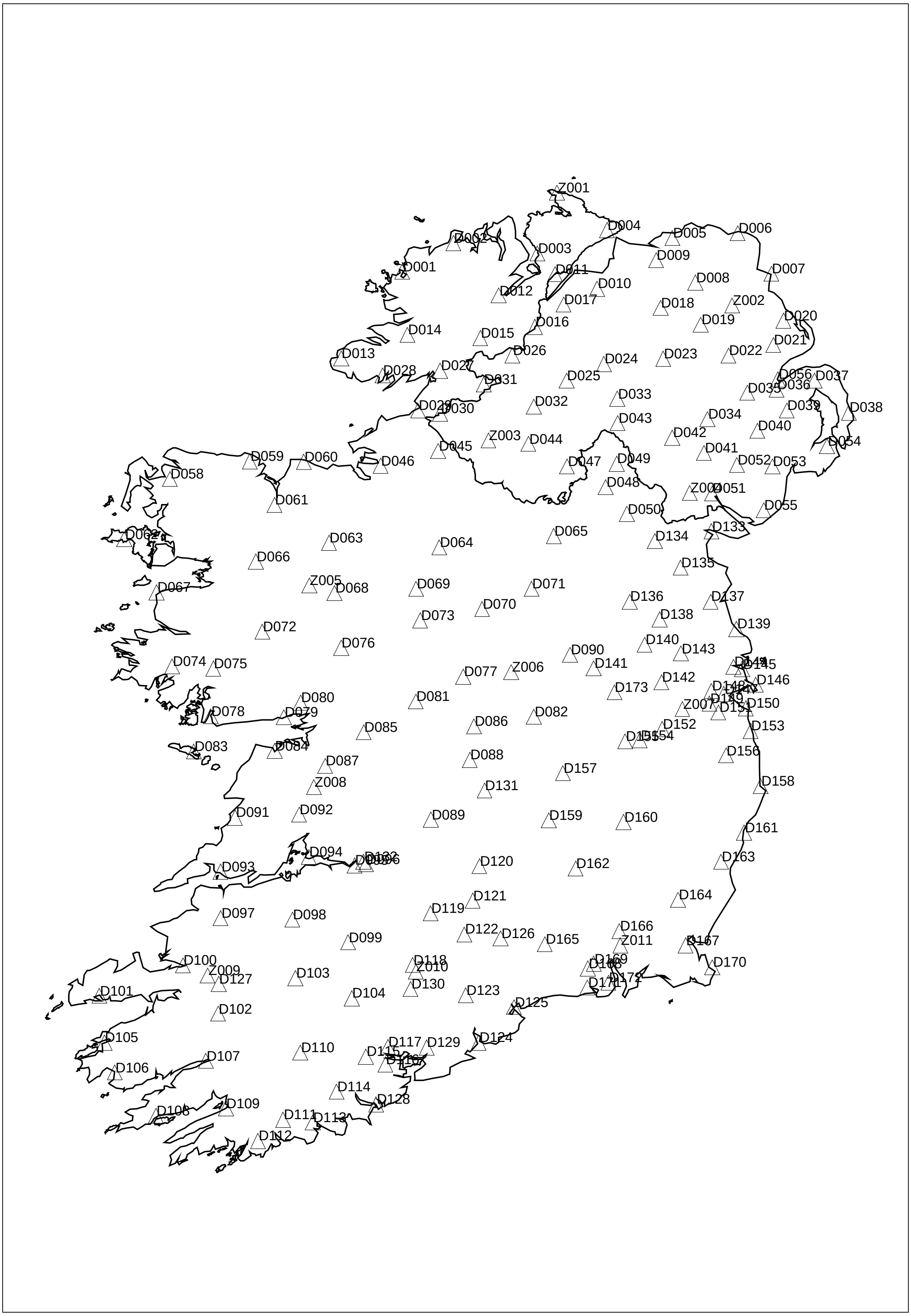

Diagram 6 : ***The Distribution of Stations***

longitude and height. For the ETRF89 co-ordinates, the height above the GRS80 ellipsoid is given. For the co-ordinates in Ireland 1975, orthometric heights (above the geoid) are given. In order to convert the orthometric heights to ellipsoidal heights, a knowledge of the geoid is required – a mean geoid-ellipsoid separation of 2.5 metres was assumed for all Ireland 1975 orthometric heights to convert them to ellipsoidal heights. However, the transformation was designed for 2 Dimensional co-ordinate transformations.

Assessment of 7 Parameter Helmert Transformation

In the following tests, a random selection of 132 (75%) stations was used for the generation of the Helmert transformation parameters, and the remaining 51 stations (25%) were used to test the algorithm by applying the parameters derived from the 75%. The final transformation parameters were determined using all data (100%).

Accuracy

In the assessment of the accuracy of the Helmert transformation an evaluation was carried out on test data (25%) based on the Root Mean Squared Error (RMSE), maximum and 95 % values of the residuals. The level of accuracy achieved is such that 95% of points transformed from one reference system are likely to be within 1 metre of their position in the other reference system and visa versa.

Invertability / Reversibility

In the assessment of the Helmert invertability / reversibility two reverse algorithms were tested. The first algorithm uses the approximate inversion of the scale and rotation matrix (as described in the paper), and the second uses an iteration algorithm. To test the invertability of the Helmert transformation, all stations in Ireland 1975 (100%) were transformed to ETRF89 and then transformed back to Ireland 1975. With the approximate inversion algorithm, the maximum difference after forward and reverse transformations was less than 0.5 mm. For the iteration algorithm, the maximum difference was less than 10^{-5} mm. Therefore the approximate inversion algorithm is suitable for most applications but the iteration algorithm is more precise.

Uniqueness

The Helmert transformation is continuous over the whole of Ireland, so there is no possibility of discrepancy between adjoining regions.

Conformality

In order to test the conformity of the Helmert transformation, the deformation of different sizes of squares were examined. Test squares of all sizes and orientation were located at all 183 stations in order to cover all regions of Ireland. It was found that the deformation caused by the Helmert transformation was very small, the maximum angle error is less than 1.8×10^{-2} arc seconds for a square up to 100 km. The offset is less than 0.3 cm for the 100 km square.

Extensibility

In order to test the extensibility of the Helmert transformation the whole set of stations was divided into two blocks. The stations within the block were used to generate the transformation parameters and the stations surrounding the block were used to test the generated transformation model. It was found that the residuals inside and outside the test block were quite similar, indicating that the Helmert transformation has very good extensibility.

Helmert 7 Parameters

The transformation parameters derived from all stations are given in Table 2 below. It should be noted that application of the Helmert 7 parameter transformation will not alter the accuracy of the mapping. Rotations are applied as radians, for conversion see Level 2 transformation example computations, Step 3.

Translations		Rotations	
ΔX (m)	+482.530	θx (")	+1.042
ΔY (m)	-130.596	θy (")	+0.214
ΔZ (m)	+564.557	θz (")	+0.631
Scale (ppm)	+8.150		

Table 2 : Helmert Transformation Parameters

Transformation Procedure

In this paper for the implementation of the Helmert transformation, 'system 1' is defined as Ireland 1975, and 'system 2' is defined as ETRF89, so that a 'forward' transformation from system 1 to system 2 converts from Ireland 1975 to ETRF89, and a reverse transformation from system 2 to system 1 converts from ETRF89 to Ireland 1975 as described in the following sections. See Appendices A, B & C for reference information.

Forward Transformation Procedure

The forward transformation procedure is used when transforming from Irish Grid co-ordinates (Easting_1, Northing_1) to ETRF89 Geodetic Ellipsoidal co-ordinates (Latitude_2, Longitude_2) and involves the following four steps:

- STEP 1: Irish Grid to Ireland 1975 Geodetic Ellipsoidal Co-ordinates
- STEP 2: Ireland 1975 Geodetic Ellipsoidal to Ireland 1975 Cartesian Co-ordinates
- STEP 3: Ireland 1975 Cartesian to ETRF89 Cartesian Co-ordinates
- STEP 4: ETRF89 Cartesian to ETRF89 Geodetic Ellipsoidal Co-ordinates outlined.

$$\mathbf{E_1, N_1 \rightarrow Lat_1, Long_1 \rightarrow X_1,Y_1,Z_1 \rightarrow \boxed{T} \rightarrow X_2,Y_2,Z_2 \rightarrow Lat_2, Long_2}$$

STEP 1: Irish Grid to Ireland 1975 Geodetic Ellipsoidal Co-ordinates

Irish Grid co-ordinates ($Easting_1$, $Northing_1$) are converted to Ireland 1975 geodetic ellipsoidal co-ordinates ($Latitude_1$, $Longitude_1$) using standard equations. For an explanation and computational example on how this conversion is implemented please see Reference [1].

STEP 2: Ireland 1975 Geodetic Ellipsoidal to Ireland 1975 Cartesian Co-ordinates

Ireland 1975 geodetic ellipsoidal co-ordinates ($Latitude_1$, $Longitude_1$) referenced to the modified Airy ellipsoid are converted to Ireland 1975 Cartesian co-ordinates (X_1, Y_1, Z_1) using the following standard equations:

$$v = \frac{a}{(1 - e^2 \sin^2 \phi_1)^{1/2}}$$

$$X_1 = (v + h) \cos \phi_1 \cos \lambda_1$$

$$Y_1 = (v + h) \cos \phi_1 \sin \lambda_1$$

$$Z_1 = (v(1 - e^2) + h) \sin \phi_1$$

Where :

- ν = Prime vertical radius of curvature
- a = Semi-major axis of the modified Airy ellipsoid (6,377,340.189 m)
- e^2 = Eccentricity squared of the modified Airy ellipsoid (0.006 670 540 15)
- ϕ_1 = Latitude (decimals of a degree)
- λ_1 = Longitude (decimals of a degree)
- h = Ellipsoidal Height (metres)

If the ellipsoidal height is unknown, it should be treated as zero in the above equations.

STEP 3: Ireland 1975 Cartesian to ETRF89 Cartesian Co-ordinates

Ireland 1975 Cartesian co-ordinates (X_1, Y_1, Z_1) are transformed to ETRF89 Cartesian co-ordinates (X_2, Y_2, Z_2) using the following 7 parameter Helmert transformation equation:

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1+\mu & \theta_z & -\theta_y \\ -\theta_z & 1+\mu & \theta_x \\ \theta_y & -\theta_x & 1+\mu \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} + \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}$$

where:

$\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}$ = Translation vector in metres between the origins of the two system

θ_x = X rotation angle in radians between the two systems

θ_y = Y rotation angle in radians between the two systems
θ_z = Z rotation angle in radians between the two systems

μ = Relative scale factor between the two systems

STEP 4: ETRF89 Cartesian to ETRF89 Geodetic Ellipsoidal Co-ordinates

ETRF89 Cartesian co-ordinates (X_2, Y_2, Z_2) are converted to ETRF89 geographic ellipsoidal co-ordinates (Latitude$_2$, Longitude$_2$) referenced to the GRS80 ellipsoid using the following standard equations:

$$\lambda_2 = \tan^{-1}\frac{Y_2}{X_2}$$

$$v = \frac{a}{(1 - e^2 \sin^2 \phi_2)^{1/2}}$$

$$\phi_2 = \tan^{-1}\frac{Z_2 + e^2 v \sin\phi_2}{(X_2{}^2 + Y_2{}^2)^{1/2}}$$

Where :

ϕ_2 = Latitude (decimals of a degree)
λ_2 = Longitude (decimals of a degree)
ν = Prime vertical radius of curvature
a = Semi-major axis of the GRS80 ellipsoid (6,378,137.000 m)
e^2 = Eccentricity squared of the GRS80 ellipsoid (0.006 694 380 022 90)

As ϕ_2 is also an unknown in the equations to compute ν and ϕ_2 an iterative process is adopted whereby the Ireland 1975 latitude value is given as the initial approximation of ϕ_2. The first iteration of ν and ϕ_2 can then be computed and the resulting ϕ_2 value used as the second approximation of ϕ_2 in the second iteration of ν and ϕ_2. Only two iterations are required to compute a final value for ϕ_2 which can be verified by the comparison of a third iteration with the second.

Reverse Transformation Procedure

The reverse transformation procedure is used when transforming from ETRF89 Geodetic Ellipsoidal co-ordinates to Irish Grid co-ordinates and involves the following four steps:

- STEP 1: ETRF89 Geodetic Ellipsoidal to ETRF89 Cartesian Co-ordinates
- STEP 2: ETRF89 Cartesian to Ireland 1975 Cartesian Co-ordinates
- STEP 3: Ireland 1975 Cartesian to Ireland 1975 Geodetic Ellipsoidal Co-ordinates
- STEP 4: Ireland 1975 Geodetic Ellipsoidal to Irish Grid Co-ordinates

Lat$_2$, Long$_2$ → X$_2$,Y$_2$,Z$_2$ → T → X$_1$,Y$_1$,Z$_1$ → Lat$_1$, Long$_1$ → E$_1$, N$_1$

STEP 1: ETRF89 Geodetic Ellipsoidal to ETRF89 Cartesian Co-ordinates

ETRF89 geodetic ellipsoidal co-ordinates (Latitude_2, Longitude_2) referenced to GRS80 ellipsoid are converted to ETRF89 Cartesian co-ordinates (X_2, Y_2, Z_2) using the following standard equations:

$$v = \frac{a}{(1 - e^2 \sin^2 \phi_2)^{1/2}}$$

$$X_2 = (v + h)\cos\phi_2 \cos\lambda_2$$

$$Y_2 = (v + h)\cos\phi_2 \sin\lambda_2$$

$$Z_2 = (v(1 - e^2) + h)\sin\phi_2$$

Where :

ν = Prime vertical radius of curvature
a = Semi-major axis of the GRS80 ellipsoid (6,378,137.000 m)
e^2 = Eccentricity squared of the GRS80 ellipsoid (0.006 694 380 022 90)
ϕ_2 = Latitude (decimals of a degree)
λ_2 = Longitude (decimals of a degree)
h = Ellipsoidal Height (metres)

STEP 2: ETRF89 Cartesian to Ireland 1975 Cartesian Co-ordinates

ETRF89 Cartesian co-ordinates (X_2, Y_2, Z_2) are transformed to Ireland 1975 Cartesian co-ordinates (X_1, Y_1, Z_1) using the following reverse 7 parameter Helmert transformation equation:

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 1-\mu & -\theta_z & \theta_y \\ \theta_z & 1-\mu & -\theta_x \\ -\theta_y & \theta_x & 1-\mu \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} - \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}$$

where:

$\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}$ = Translation vector in metres between the origins of the two system

θ_x = X rotation angle in radians between the two systems
θ_y = Y rotation angle in radians between the two systems
θ_z = Z rotation angle in radians between the two systems

μ = Relative scale factor between the two systems

STEP 3: Ireland 1975 Cartesian to Ireland 1975 Geodetic Ellipsoidal Co-ordinates

Ireland 1975 Cartesian co-ordinates (X_1, Y_1, Z_1) are converted to Ireland 1975 geographic ellipsoidal co-ordinates (Latitude$_1$, Longitude$_1$) referenced to the modified Airy ellipsoid using the following standard equations:

$$\lambda_1 = \tan^{-1} \frac{Y_1}{X_1}$$

$$\nu = \frac{a}{(1 - e^2 \sin^2 \phi_1)^{1/2}}$$

$$\phi_1 = \tan^{-1} \frac{Z_1 + e^2 \nu \sin \phi_1}{(X_1{}^2 + Y_1{}^2)^{1/2}}$$

Where :

ϕ_1 = Latitude (decimals of a degree)
λ_1 = Longitude (decimals of a degree)
ν = Prime vertical radius of curvature
a = Semi-major axis of the modified Airy ellipsoid (6,377,340.189 m)
e^2 = Eccentricity squared of the modified Airy ellipsoid (0.006 670 540 15)

As ϕ_1 is also an unknown in the equations to compute ν and ϕ_1 an iterative process is adopted whereby the ETRF89 latitude value is given as the initial approximation of ϕ_1. The first iteration of ν and ϕ_1 can then be computed and the resulting ϕ_1 value used as the second approximation of ϕ_1 in the second iteration of ν and ϕ_1. Only two iterations are required to compute a final value for ϕ_1 which can be verified by the comparison of a third iteration with the second.

STEP 4: Ireland 1975 Geodetic Ellipsoidal to Irish Grid Co-ordinates

Ireland 1975 geodetic ellipsoidal co-ordinates (Latitude$_1$, Longitude$_1$) are converted to Irish Grid co-ordinates (Easting$_1$, Northing$_1$) using standard equations. For an explanation and computational example on how this conversion is implemented please see Reference [1].

EXAMPLE COMPUTATIONS

Level 1 Transformation (Forward Case)

STEP 1: Irish Grid Co-ordinates converted to GPS (Irish Grid) Co-ordinates

	Easting (m)	Northing (m)
Irish Grid ($East_1$, $North_1$)	271,707.4	248,879.6
Shifts	-49.0	+23.4
GPS (Irish Grid)	271,658.4	248,903.0

STEP 2: GPS (Irish Grid) converted to ETRF89 Geodetic Ellipsoidal Co-ordinates

Latitude (ϕ_2)	Longitude (λ_2)
53º 29’ 06”.96840	-6º 55’ 13”.92478

Level 1 Transformation (Reverse Case)

STEP 1: ETRF89 Geodetic Ellipsoidal Co-ordinates projected to GPS (Irish Grid)

Latitude (ϕ_2)	Longitude (λ_2)
53º 29’ 06”.96840	-6º 55’ 13”.92478

Easting	Northing
271,658.4 m	248,903.0 m

STEP 2: GPS (Irish Grid) Co-ordinates converted to Irish Grid Co-ordinates

	Easting (m)	Northing (m)
GPS (Irish Grid)	271,658.4	248,903.0
Shifts	+49.0	-23.4
Irish Grid ($East_1$, $North_1$)	271,707.4	248,879.6

Level 2 Transformation (Forward Case)

STEP 1: Irish Grid to Ireland 1975 Geodetic Ellipsoidal Co-ordinates

$Easting_1$	=	271,707.427 m
$Northing_1$	=	248,879.641 m
ϕ_1 = 53º 29’ 06”.17996	=	53º.485049988889
λ_1 = -6º 55’ 10”.77000	=	-6º.919658333333

STEP 2: Ireland 1975 Geodetic Ellipsoidal to Ireland 1975 Cartesian Co-ordinates

h_1 = 0.000 m		(Set to zero as height is unknown)
$v = \dfrac{a}{(1-e^2 \sin^2 \phi_1)^{1/2}}$	=	6,391,123.911284 m
$X_1 = (v+h)\cos\phi_1 \cos\lambda_1$	=	3,775,226.258140 m
$Y_1 = (v+h)\cos\phi_1 \sin\lambda_1$	=	-458,166.888768 m
$Z_1 = (v(1-e^2)+h)\sin\phi_1$	=	5,102,293.084465 m

STEP 3: Ireland 1975 Cartesian to ETRF89 Cartesian Co-ordinates

θ_x = 1”.042	=	0.0000050518 Radians
θ_y = 0”.214	=	0.0000010375 Radians
θ_z = 0”.631	=	0.0000030592 Radians

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1+\mu & \theta_z & -\theta_y \\ -\theta_z & 1+\mu & \theta_x \\ \theta_y & -\theta_x & 1+\mu \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} + \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}$$

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1.0000081500 & 0.0000030592 & -0.0000010375 \\ -0.0000030592 & 1.0000081500 & 0.0000050518 \\ 0.0000010375 & -0.0000050518 & 1.0000081500 \end{bmatrix} \begin{bmatrix} 3775226.258140 \\ -458166.888768 \\ 5102293.084465 \end{bmatrix} + \begin{bmatrix} 482.530 \\ -130.596 \\ 564.557 \end{bmatrix}$$

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 3775250.330986 \\ -458156.396351 \\ 5102340.899504 \end{bmatrix} + \begin{bmatrix} 482.530 \\ -130.596 \\ 564.557 \end{bmatrix}$$

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 3775732.860986 \\ -458286.992351 \\ 5102905.456504 \end{bmatrix}$$

X_2 = 3,775,732.860986 m

Y_2 = -458,286.992351 m

Z_2 = 5,102,905.456504 m

STEP 4: ETRF89 Cartesian to ETRF89 Geodetic Ellipsoidal Co-ordinates

$$\lambda_2 = \tan^{-1} \frac{Y_2}{X_2} \quad = \quad -6^\circ.9205349866$$

$$v = \frac{a}{(1 - e^2 \sin^2 \phi_2)^{1/2}}$$

$$\phi_2 = \tan^{-1} \frac{Z_2 + e^2 v \sin \phi_2}{(X_2{}^2 + Y_2{}^2)^{1/2}}$$

Iteration No.1 — ν = 6,391,971.872757 m
ϕ_2 = 53º.4852663624

Iteration No.2 — ν = 6,391,971.950371 m
ϕ_2 = 53º.4852668774

Iteration No.3 — ν = 6,391,971.950556 m
ϕ_2 = 53º.4852668787

Iteration No.4 — ν = 6,391,971.950557 m
ϕ_2 = 53º.4852668787

ϕ_2 = 53º 29’ 06”.96076

λ_2 = -6º 55’ 13”.92595

Level 2 Transformation (Reverse Case)

STEP 1: ETRF89 Geodetic Ellipsoidal to ETRF89 Cartesian Co-ordinates

ϕ_2 = 53º 29’ 06”.96076 = 53º.485266877778

λ_2 = -6º 55’ 13”.92595 = -6º.920534986111

h_2[1] = 125.355 m

$$v = \frac{a}{(1 - e^2 \sin^2 \phi_2)^{1/2}} = 6{,}391{,}971.950556 \text{ m}$$

$$X_2 = (v + h)\cos\phi_2 \cos\lambda_2 = 3{,}775{,}774.923481 \text{ m}$$

$$Y_2 = (v + h)\cos\phi_2 \sin\lambda_2 = -458{,}292.097739 \text{ m}$$

$$Z_2 = (v(1 - e^2) + h)\sin\phi_2 = 5{,}102{,}962.686942 \text{ m}$$

STEP 2: ETRF89 Cartesian to Ireland 1975 Cartesian Co-ordinates

θ_x = 1”.042 = 0.0000050518 Radians

θ_y = 0”.214 = 0.0000010375 Radians

θ_z = 0”.631 = 0.0000030592 Radians

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 1-\mu & -\theta_z & \theta_y \\ \theta_z & 1-\mu & -\theta_x \\ -\theta_y & \theta_x & 1-\mu \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} - \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 0.9999918500 & -0.0000030592 & 0.0000010375 \\ 0.0000030592 & 0.9999918500 & -0.0000050518 \\ -0.0000010375 & 0.0000050518 & 0.9999918500 \end{bmatrix} \begin{bmatrix} 3775774.923481 \\ -458292.097739 \\ 5102962.686942 \end{bmatrix} - \begin{bmatrix} 482.530 \\ -130.596 \\ 564.557 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 3775750.847241 \\ -458302.590840 \\ 5102914.865243 \end{bmatrix} - \begin{bmatrix} 482.530 \\ -130.596 \\ 564.557 \end{bmatrix}$$

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[1] The Ellipsoidal Height h_2 (height above the GRS80 Reference Ellipsoid) is included in the level 2 reverse case transformation as users will be able to obtain this value from their GPS equipment.
However, the transformation was designed for 2 Dimensional co-ordinate transformations.

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 3775268.317241 \\ -458171.994840 \\ 5102350.308243 \end{bmatrix}$$

X_1 = 3,775,268.317241 m

Y_1 = -458,171.994840 m

Z_1 = 5,102,350.308243 m

STEP 3: Ireland 1975 Cartesian to Ireland 1975 Geodetic Ellipsoidal Co-ordinates

$$\lambda_1 = \tan^{-1}\frac{Y_1}{X_1} \qquad = \qquad -6^{\circ}.9196583590$$

$$\nu = \frac{a}{(1 - e^2 \sin^2 \phi_1)^{1/2}}$$

$$\phi_1 = \tan^{-1}\frac{Z_1 + e^2 \nu \sin \phi_1}{({X_1}^2 + {Y_1}^2)^{1/2}}$$

Iteration No.1 ν = 6,391,123.988795 m
ϕ_1 = 53º.4850504930

Iteration No.2 ν = 6,391,123.911465 m
ϕ_1 = 53º.4850499798

Iteration No.3 ν = 6,391,123.911281 m
ϕ_1 = 53º.4850499785

Iteration No.4 ν = 6,391,123.911281 m
ϕ_1 = 53º.4850499785

ϕ_1 = 53º 29’ 06”.17992

λ_1 = -6º 55’ 10”.77009

STEP 4: Ireland 1975 Geodetic Ellipsoidal to Irish Grid Co-ordinates

Easting_1 = 271,707.425 m

Northing_1 = 248,879.640 m

REFERENCES

[1] Ordnance Survey Ireland & Ordnance Survey of Northern Ireland, 1999. *The Irish Grid.* OS*i* Dublin, OSNI Belfast.

[2] Allan, A.L., 1993. *Practical Surveying and Computations.* 2nd Edition, Butterworth/Heinemann Limited.

[3] Moore, T., Smith, M. J., 1998. *Geodetic Transformations – Part 1.* Survey Review, Vol 34 No 269, July 1998.

[4] Moore, T., Smith, M. J., 1998. *Geodetic Transformations – Part 2.* Survey Review, Vol 34 No 270, October 1998.

[5] Cory, M.J., 1997. *Re-measuring the size of Ireland.* Survey Ireland.

[6] Bomford G., 1980. *Geodesy*. 4th Edition, Oxford University Press.

APPENDIX A

NOTATION, SYMBOLS AND STANDARD FORMULAE

All distances are in metres :
    Conversion feet to metres: 1 ft = 0.304 800 749 1 m

All angles are in radians
    Conversion degrees (decimal) to radians $1^{o} = \frac{\pi}{180}$, or 0.017 453 293 radians

Notation	Description, Formulae and Constants
a	Semi-major axis of the reference ellipsoid.
b	Semi-minor axis of the reference ellipsoid. $b = \sqrt{a^2\left(1-e^2\right)}$
e^2	Eccentricity squared of the reference ellipsoid. $e^2 = \frac{a^2 - b^2}{a^2}$
v	Prime vertical radius of curvature. $v = \frac{a}{(1 - e^2 \sin^2 \phi)^{1/2}}$
ϕ	Latitude of a point. An iterative process is used to in this calculation (see pages 9 and 11) $\phi = \tan^{-1} \frac{Z + e^2 v \sin\phi}{(X^2 + Y^2)^{1/2}}$
λ	Longitude of a point (positive (+) east of Greenwich and negative (-) west of Greenwich). $\lambda = \tan^{-1} \frac{Y}{X}$
h	Height of a point above the reference ellipsoid

X	X co-ordinate of a point in a Cartesian co-ordinate reference system
Y	Y co-ordinate of a point in a Cartesian co-ordinate reference system.
Z	Z co-ordinate of a point in a Cartesian co-ordinate reference system
$\begin{bmatrix}\Delta X \\ \Delta Y \\ \Delta Z\end{bmatrix}$	Helmert transformation translation vector in metres between the origins of two Cartesian co-ordinate reference systems.
θ_x θ_y θ_z	Helmert transformation rotation parameters in radians between two Cartesian co-ordinate reference systems.
μ	Helmert transformation relative scale factor in parts per million (ppm) between two Cartesian co-ordinate reference systems.

APPENDIX B

REFERENCE ELLIPSOIDS

REFERENCE ELLIPSOID	DEFINING PARAMETERS	COMMENTS
Airy	a = 6 377 563.3964 m b = 6 356 256.9096 e^2 = 0.006 670 540 000 12	
GRS80	a = 6 378 137.000 m b = 6 356 752.314 1 m e^2 = 0.006 694 380 022 90	Strictly speaking GRS80 is defined by many more constants, however these are not of direct interest in this instance.[2]
WGS84	a = 6 378 137.000 m e^2 = 0.006 694 379 9	There are other defining parameters.[3]

[2] Moritz, H., 1988. 'G*eodetic Reference System 1980'*. Bulletin Geodesique, 1988 Volume 62 No 3, Paris.
[3] Defence Mapping Agency, 1987. *Department of Defence World Geodetic System 1984. Technical Report (and supplements).* DMA TR-8350.2, USA.

APPENDIX C

IRISH GRID PARAMETERS

National Reference System	Irish Grid
Reference Ellipsoid	Airy Modified
Geodetic Datum	1965 Datum
Vertical Datum	Malin Head
Map Projection	Transverse Mercator
Measurement Unit	International metre

Note: Heights in Northern Ireland are related to Belfast Datum

THE TRANSVERSE MERCATOR MAP PROJECTION

Ellipsoid	Airy Modified
True Origin	Latitude 53^{o} 30’ 00” N Longitude 8^{o} 00’ 00” W
False Origin	200 kms west of true origin 250 kms south of true origin
Plane Co-ordinates of True Origin	200 000 E 250 000 N
Scale Factor on Central Meridian	1.000 035

APPENDIX D

GPS MANUFACTURERS AND THE LEVEL 2 TRANSFORMATION

Ashtech

Ashtech
11, Blenheim Office Park, Lower Road, Long Hanborough, Oxfordshire, OX 88LN, England.
Phone: [+44-1993] 883 533

Hempenstall Survey Limited
56 Landsdowne Road, Ballsbridge, Dublin 4, Ireland.
Phone: [+353-1] 668 8170

Geotronics

Geodata Limited
15 Riverview Business Park, New Nangor Road, Dublin 12, Ireland.
Phone: [+353-1] 460 4404

Leica (Wild)

Leica Geosystems Limited
Davy Avenue, Knowlhill, Milton keynes, MK5 8LB, England.
Phone: [+44-1908] 256 500

Survey Instruments Services
Unit 6A, Ballymount Cross Industrial Estate, Dublin 24, Ireland.
Phone: [+353-1] 456 8659

Magellan

Positioning Resources Limited
64 Commerce Street, Aberdeen, AB11 5FP, Scotland.
Phone: [+44-1224] 581 502

Hempenstall Survey Limited
56 Landsdowne Road, Ballsbridge, Dublin 4, Ireland.
Phone: [+353-1] 668 8170

Omnistar

Positioning Resources Limited
64 Commerce Street, Aberdeen, AB11 5FP, Scotland.
Phone: [+44-1224] 581 502

Sokkia

Sokkia
Datum House, Electra Way, Crewe Business Park, Crewe, Cheshire, CW1 6ZT, England.
Phone: [+44-1270] 250 525

Celtic Surveys Limited
11 Berkley Street, Dublin 11, Ireland.
Phone: [+353-1} 830 4855

Topcon

Topcon GB Limited
Unit 6, Oldstation Close, Coalville, Leicestershire, LE 67 3FM, England.
Phone: [+44-1530] 813 648

Topcon Ireland Limited
Unit 56, Western Parkway, Business Centre, Lower Ballymount Road, Dublin 12, Ireland.
Phone: [+353-1] 460 0021

Trimble

Trimble Navigation Limited
Trimble House, Meridian Office Park, Osborn Way, Hook, Hampshire, RG27 9HX, England.
Phone: [+44-1256] 760 150